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Semi-simple Lie Algebras and Applications

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Topics covered

- General notions of Lie groups and algebras
- Classical groups as metrics preserving groups;
- Typical parametrization of classical groups
- Applications in physics

Plan

1. Basics of simple Lie algebras
2. Root systems, Cartan-Weyl basis
3. Dynkin diagrams, fundamental weights, finite dimensional representations.
4. Automorphisms of finite order, Weyl group, gradings
5. Integrable systems and Lax representations
6. Simple Lie algebras and integrable systems.
7. Classical R-matrix and classical Yang-Baxter eq.

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LECTURE 1
General notions of Lie groups and algebras

Definition: $\mathcal{G}$ is a Lie group if its elements T_i satisfy:

1. $T_i T_j = T_k \in \mathcal{G}$;
2. $(T_1 T_2) T_3 = T_1 (T_2 T_3)$ associativity
3. there exist $\mathbb{1}$ such that $\mathbb{1} T_i = T_i$;
4. there exist T_i^{-1} such that $T_i T_i^{-1} = T_i^{-1} T_i = \mathbb{1}$.

Examples: Finite groups:

$$\mathbb{Z}_2 \quad : \quad \mathbb{1}, T_1, \qquad T_1^2 = \mathbb{1}$$

$$\mathbb{1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad T_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad T_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad \cdots$$

Each realization of T_i is called representation.

$$\mathbb{Z}_h \quad : \quad \{1\!\!1, T_1, T_1^2, \ldots, T_1^{h-1}\}, \qquad T_1^h = 1\!\!1$$

$$h = 3, \qquad 1\!\!1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad T_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

The permutation group $\mathcal{S}_n$, $n = 5$:

$$T_1 \colon (1,2,3,4,5) \to (2,1,3,4,5), \qquad T_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

$$T_2 \colon (1,2,3,4,5) \to (2,3,4,5,1), \qquad T_2 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

Continuous groups – coordinate transformations

Rotation on the plane $SO(2)$: $$T(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \qquad T(\theta_1)T(\theta_2) = T(\theta_1+\theta_2),$$

$$T(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad T^{-1}(\theta) = T(-\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix},$$

This group is commutative.

Rotation in the 3-dim space (Eiler angles):

$$T_1(\psi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & \sin\psi \\ 0 & -\sin\psi & \cos\psi \end{pmatrix}, \; T_2(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}, \; T_3(\phi) = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

This group $SO(3)$ is non-commutative, i.e. $T_1(\phi)T_2(\theta) \neq T_2(\theta)T_1(\phi)$.

Importance due to numerous applications in:

- solid state physics (expecially cristals);
- systems of ordinary differential equations (linear changes of variables)
- differential geometry, gravitation
- nuclear theory
- quantum mechanics
- elementary particle theory
- Hamiltonian systems
- conformal field theories, integrable models

The Lie groups $\mathcal{G}$ are nonlinear manifolds.
The exponential mapping relates them to Lie algebras $\mathfrak{g}$, which are linear manifolds.

$$T_i(t) = \exp(tX_i), \qquad X_i \quad \text{is the generator of } \mathfrak{g}.$$

$$T_i(t) = \mathbb{1} + tX_i + \frac{1}{2}t^2X_i^2 + \cdots, \qquad X_i = \left.\frac{dT_i}{dt}\right|_{t=0}.$$

The group commutator:

$$\begin{aligned}
&T_i(t)T_j(t)T_i^{-1}(t)T_j^{-1}(t) \\
&= (\mathbb{1} + X_it + \cdots)(\mathbb{1} + X_jt + \cdots)(\mathbb{1} - X_it + \cdots)(\mathbb{1} - X_jt + \cdots) \\
&= \mathbb{1} + t^2(X_iX_j - X_jX_i) + \cdots \\
&= \mathbb{1} + t^2[X_i, X_j] + \cdots
\end{aligned}$$

Lie bracket:

$$[X_i, X_j] = X_iX_j - X_jX_i.$$

From now on we will view the elements of $\mathcal{G}$ and $\mathfrak{g}$ as matrices. The group multiplication is just the multiplication of matrices.

Along with this realization there are many others in terms of

- partial differential operators
- vector fields
- creation and annihilation operators

Indeed, let us introduce:

$$E_{kj} = \begin{pmatrix} 0 & \dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & & \vdots \\ 0 & \dots & 1 & \dots & 0 \\ \vdots & & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \dots & 0 \end{pmatrix} \begin{matrix} \\ \\ k \\ \\ \\ \end{matrix}$$
$$\qquad\qquad\qquad j$$

i.e.

$$(E_{kj})_{lm} = \delta_{kl}\delta_{jm}, \qquad \mathcal{D}_{kj} = x_k \frac{\partial}{\partial x_j}, \qquad \mathcal{A}_{kj} = a_k^\dagger a_j,$$

where $a_k^\dagger, a_j$ are the creation and annihilation operators satisfying the canonical commutation relations:

$$[a_k^\dagger, a_j^\dagger] = 0, \qquad [a_k^\dagger, a_j] = \delta_{kj}, \qquad [a_k, a_j] = 0.$$

Then it is easy to check that:

$$E_{kj}\,, \qquad \mathcal{A}_{kj}\,, \qquad \mathcal{D}_{kj}$$

satisfy the same commutation relations:

$$[E_{kj}\,, E_{sp}] = \delta_{js} E_{kp} - \delta_{kp} E_{sj}\,, \quad [\mathcal{D}_{kj}\,, \mathcal{D}_{sp}] = \delta_{js} \mathcal{D}_{kp} - \delta_{kp} \mathcal{D}_{sj}\,,$$

$$[\mathcal{A}_{kj}\,, \mathcal{A}_{sp}] = \delta_{js} \mathcal{A}_{kp} - \delta_{kp} \mathcal{A}_{sj}\,.$$

If we have a matrix representation of a Lie algebra, we can immediately construct a representation in terms of vector fields, or in creation-annihilation operators.

The generators of $so(3)$:

$$X_1 = \frac{dT_1}{d\psi}\bigg|_{\psi=0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \qquad X_2 = \frac{dT_2}{d\theta}\bigg|_{\theta=0} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

$$X_3 = \frac{dT_3}{d\phi}\bigg|_{\phi=0} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The commutation relations of $so(3)$:

$$[X_1, X_2] = X_3, \quad [X_2, X_3] = X_1, \quad [X_3, X_1] = X_2, \text{ or } \quad [X_i, X_j] = \epsilon_{ijk} X_k$$

where ϵ_{ijk} is the completely antisymmetric tensor $\epsilon_{123} = 1$.

Representation as vector fields:

$$M_1 = x_1 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_1}, \quad M_2 = x_1 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_1}, \quad M_3 = x_2 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_2}.$$

or in terms of creation and annihilation operators:

$$S_1 = a_1^\dagger a_2 - a_2^\dagger a_1, \quad S_2 = a_1^\dagger a_3 - a_3^\dagger a_1, \quad S_3 = a_2^\dagger a_3 - a_3^\dagger a_2.$$

Obviously

$$[M_1, M_2] = M_3, \quad [M_2, M_3] = M_1, \quad [M_3, M_1] = M_2, \text{ or } \quad [M_i, M_j] = \epsilon_{ijk} M_k$$

$$[S_1, S_2] = S_3, \quad [S_2, S_3] = S_1, \quad [S_3, S_1] = S_2, \text{ or } \quad [S_i, S_j] = \epsilon_{ijk} S_k$$

Classical Lie groups as transformations preserving metrics in linear spaces $\mathcal{V}$:

- **Orthogonal groups** $O(n)$. $\mathcal{V} \simeq \mathbb{E}^n$ with metric generated by the scalar product

$$(\vec{x}, \vec{y}) \equiv \vec{x}^T \vec{y} = \sum_{j=1}^{n} x_j y_j .$$

Then any element $T \in O(n)$ is defined as:

$$(T\vec{x}, T\vec{y}) \equiv (T\vec{x})^T T\vec{y} = \vec{x}^T T^T T\vec{y} \equiv \vec{x}^T \vec{y} \qquad \leftrightarrow \qquad T^T T = \mathbb{1} .$$

For the algebra elements: $\quad X + X^T = 0$.

- **Unitary groups** $U(n)$. $\mathcal{V} \simeq \mathbb{C}^n$ with metric generated by the scalar product

$$(\vec{x}, \vec{y}) \equiv \vec{x}^\dagger \vec{y} = \sum_{j=1}^{n} x_j^* y_j .$$

Then any element $T \in U(n)$ is defined as:

$$(T\vec{x}, T\vec{y}) \equiv (T\vec{x})^\dagger T\vec{y} = \vec{x}^\dagger T^\dagger T\vec{y} \equiv \vec{x}^\dagger \vec{y} \qquad \leftrightarrow \qquad T^\dagger T = \mathbb{1} .$$

For the algebra elements: $X + X^\dagger = 0.$

- **Symplectic groups** $SP(2n)$. $\mathcal{V} \simeq \mathbb{E}^{2n}$ or $\mathbb{C}^{2n}$ with metric generated by the skew-scalar product

$$(\vec{x}, \vec{y}) \equiv \vec{x}^T S_1 \vec{y} = \sum_{j=1}^{n} (x_j y_{n+j} - x_{n+j} y_j), \qquad S_1 = \begin{pmatrix} 0 & 1\!\!1 \\ -1\!\!1 & 0 \end{pmatrix}.$$

Then any element $T \in SP(2n)$ is defined as:

$$(T\vec{x}, T\vec{y}) \equiv (T\vec{x})^T S_1 T\vec{y} = \vec{x}^T T^T S_1 T\vec{y} \equiv (\vec{x}^T, \vec{y}) \qquad \leftrightarrow \qquad T^T S_1 T = S_1.$$

For the algebra elements: $S_1 X + X^T S_1 = 0.$

- **The special** orthogonal $SO(n)$, special unitary $SU(n)$ and special symplectic groups: request in addition

$$\det T(t) \equiv \exp(tX) = 1, \qquad \text{therefore} \qquad \text{tr}\, X = 0.$$

Proof: Let τ_j be the eigenvalues of T. Then

$$T = \mathcal{U}^{-1} \begin{pmatrix} \tau_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \tau_n \end{pmatrix} \mathcal{U} = \mathcal{U}^{-1} \exp \begin{pmatrix} \ln \tau_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \ln \tau_n \end{pmatrix} \mathcal{U}.$$

If $\det T \equiv \prod_{j=1}^{n} \tau_j = 1$ then

$$\text{tr}\, X = \sum_{j=1}^{n} \ln \tau_j = \ln \prod_{j=1}^{n} \tau_j = \ln 1 = 0.$$

Lie algebras:

The Lie algebra $\mathfrak{g}$ is a linear space equipped with a Lie bracket $[\cdot\ ,\ \cdot]$ having the properties:

- **Linearity:** $[c_1 X + c_2 Y, Z] = c_1 [X, Z] + c_2 [Y, Z]$;
- **skew-symmetricity:** $[X, Y] = -[Y, X]$;

- **Jacobi identity:** $[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]=0.$

Define the structure constants C^l_{jk} of $\mathfrak{g}$:

Assume X_j, $j=1,\dots,L=\dim\mathfrak{g}$ form a basis in $\mathfrak{g}$. Then

$$[X_j,X_k]=\sum_{l=1}^{L}C^l_{jk}X_l.$$

- **Properties of the structure constants:**
  skew-symmetricity: $C^l_{jk}=-C^l_{kj}$;
  Jacobi identity: $\sum_{s=1}^{L}(C^p_{js}C^s_{kl}+C^p_{ks}C^s_{lj}+C^p_{ls}C^s_{jk}=0.$

- **Remark:** Any element $X\in\mathfrak{g}$ may be written as $X=\sum_{j=1}^{L}c_jX_j$. The constants c_j can be either real or complex. So we can consider $\mathfrak{g}$ over the real or complex numbers. In what follows c_j will be considered complex.

Definition 1 (Homomorphism) — *mapping of one algebra $\mathfrak{g}$ to another $\mathfrak{g}'$ which is linear and preserves the algebraic operation:*

$$p:\mathfrak{g}\to\mathfrak{g}',\qquad \begin{aligned} p(\alpha X+\beta Y) &= \alpha p(X)+\beta p(Y),\\ p([X,Y]) &= [p(X),p(Y)]; \end{aligned} \tag{1}$$

Definition 2 (Isomorphism) *— a homomorphism p which is one to one.*

For example the algebras $su(2)$ and $so(3)$ are isomorphic, while their groups are not isomorphic. Also $so(4)$ and $so(3) \otimes so(3)$.

Definition 3 (Automorphisms) *— $p : \mathfrak{g} \to g$ with p – one to one.*

$$\begin{aligned} p(\alpha X + \beta Y) &= \alpha p(X) + \beta p(Y), \\ p([X, Y]) &= [p(X), p(Y)]. \end{aligned} \tag{2}$$

Example 1 $p(X) = AXA^{-1}$*, where $A \in \mathcal{G}$. This an inner automorphism. All other automorphism are outer automorphisms. For example:*

$$\varphi(X) = -X^t. \tag{3}$$

If $\varphi^2 \equiv \mathbf{id}$, i.e. if $\varphi(\varphi(X)) \equiv X$, then φ is an involutive automorphism. Other examples of such automorphisms are provided by:

$$\varphi(X) = -X^\dagger, \qquad \varphi(X) = X^*. \tag{4}$$

Definition 4 (Subalgebra) *— a subset $\mathfrak{g}_0$ of a Lie algebra $\mathfrak{g}$ is called a subalgebra of $\mathfrak{g}$ if:*

- $\mathfrak{g}_0$ *is a linear subspace of* $\mathfrak{g}$*;*
- $[X, Y] \in \mathfrak{g}_0$ *for all* $X, Y \in \mathfrak{g}_0$*.*

Definition 5 *The subalgebra is abelian if* $[X, Y] = 0$ *for all* $X, Y \in \mathfrak{g}_0$*.*

Definition 6 (Ideal) *— a subset* $\mathfrak{g}_0 \subset \mathfrak{g}$ *forms an ideal or invariant subalgebra of* $\mathfrak{g}$ *if:*

- $\mathfrak{g}_0$ *is a linear subspace of* $\mathfrak{g}$*;*
- $[X, Y] \in \mathfrak{g}_0$ *for any* $X \in \mathfrak{g}_0$ *and* $Y \in \mathfrak{g}$*.*

Obviously $\mathfrak{g}$ is an ideal to itself and $\{0\}$ is an ideal to any Lie algebra $\mathfrak{g}$. If $\mathfrak{g}$ contains elements that are not in $\mathfrak{g}_0$ and $\mathfrak{g}_0 \not\equiv \{0\}$, then $\mathfrak{g}_0$ is a *proper* ideal.

Example 2 $sl(n, \mathbb{C})$ *is an ideal of* $gl(n, \mathbb{C})$*;*

- $so(n, \mathbb{C})$ *is an ideal of* $o(n, \mathbb{C})$*;*
- $o(n, \mathbb{C})$ *is a subalgebra of* $gl(n, \mathbb{C})$ *but is* not *an ideal of* $gl(n, \mathbb{C})$*.*

Theorem 1 ([Ado]) *. Every Lie algebra allows matrix representation.*

Adjoint Action and Adjoint Representation. To each element $X \in \mathfrak{g}$ we can put into correspondence the linear operator $\mathrm{ad}(X)$:

$$\mathrm{ad}_X = [X, \cdot], \qquad \text{i.e.} \qquad \mathrm{ad}_X \cdot Z = [X, Z]. \tag{5}$$

Prove that

$$[\mathrm{ad}_X, \mathrm{ad}_Y] \cdot = \mathrm{ad}_{[X,Y]} \cdot \tag{6}$$

Indeed, start with the Jacobi identity:

$$[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0,$$

$$\text{ad}_X \cdot \text{ad}_Y Z - \text{ad}_{[X,Y]} Z - \text{ad}_Y \text{ad}_X Z = 0,$$

q.e.d.

From the definitions (5) we have:

$$\mathrm{ad}_{X_k} \cdot X_l = [X_k, X_l] = \sum_m C_{kl}^m X_m. \tag{7}$$

Thus ad_{X_k} may be viewed as an $L \times L$ matrix with matrix elements given by the structure constants:

$$(\mathrm{ad}_{X_k})_{ml} = C^m_{kl}. \tag{8}$$

Then using Jacobi identity for C^m_{kl} one can prove:

$$[\mathrm{ad}_{X_k}, \mathrm{ad}_{X_m}] \cdot = \sum_p C^p_{kl} \mathrm{ad}_{X_p} \cdot \tag{9}$$

Definition 7 *The Killing form:*

$$B(X, Y) = \mathrm{tr}\,(\mathrm{ad}_X \cdot \mathrm{ad}_Y \cdot) \tag{10}$$

is a symmetric bilinear form on $\mathfrak{g}$. In particular:

$$B(X_k, X_p) = \mathrm{tr}\left(\mathrm{ad}_{X_k} \cdot \mathrm{ad}_{X_p} \cdot\right) = \sum_{l,m} C^m_{kl} C^l_{pm} = g_{kp}, \tag{11}$$

give the components of the Killing tensor*.*

Theorem 2 *The Killing form has two important properties:*

1. *It is independent of the choice of the basis in $\mathfrak{g}$;*
2. *It is invariant under the group of automorphisms of $\mathfrak{g}$.*

Proof Let $X, Y \in \mathfrak{g}$ and let

$$X = \sum_k x_k X_k, \qquad Y = \sum_l y_l X_l. \tag{12}$$

Then

$$B(X,Y) = \sum_{k,l} x_k y_l B(X_k, X_l) = \sum_{k,l} x_k y_l g_{k,l}. \tag{13}$$

Let us consider the commutation relations:

$$[X_k, X_l] = \sum_m C_{kl}{}^m X_m \tag{14}$$

and let us apply to both sides the automorphism p. Using the main property we get:

$$[p(X_k), p(X_l)] = \sum_m C_{kl}{}^m p(X_m) \tag{15}$$

i.e., the automorphisms do not change the structure constants ${C_{kl}}^m$. Therefore:

$$B\left(p(X_k), p(X_l)\right) = \sum_{l,m} {C_{kl}}^m {C_{pm}}^l = B\left(X_k, X_l\right). \tag{16}$$

This invariance of $B(X,Y)$ has the following consequences. Indeed, let p be an automorphism of the form:

$$p(X) = AXA^{-1}, \qquad A = \exp(\alpha Z), \qquad Z \in \mathfrak{g}. \tag{17}$$

Suppose also, that α is small; then:

$$\begin{aligned} p(X) &= X + \alpha[Z,X] + \mathcal{O}\left(\alpha^2\right), \\ p(Y) &= Y + \alpha[Z,Y] + \mathcal{O}\left(\alpha^2\right). \end{aligned} \tag{18}$$

Inserting these relations into $B(p(X), p(Y))$ we get the expression:

$$\begin{aligned} &B\left(X + \alpha[Z,X] + \mathcal{O}\left(\alpha^2\right), Y + \alpha[Z,Y] + \mathcal{O}\left(\alpha^2\right)\right) \\ &= B(X,Y) + \alpha\left(B\left([Z,X],Y\right) + B\left(X,[Z,Y]\right)\right) + \mathcal{O}\left(\alpha^2\right) \end{aligned} \tag{19}$$

which must coincide with $B(X,Y)$. Therefore:

$$B([Z,X],Y) + B(X,[Z,Y]) = 0, \tag{20}$$

must hold identically for any choice of $X, Y, Z \in \mathfrak{g}$. This relation is known as local formulation of the invariance of $B(X,Y)$ under the group of automorphisms of $\mathfrak{g}$.

Definition 8 $\mathfrak{g}$ *is simple if it contains no proper ideals;*

Definition 9 $\mathfrak{g}$ *is semisimple if it contains no abelian ideals except* $\{0\}$*;*

Theorem 3 $\mathfrak{g}$ *is semisimple if and only if*

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_s \tag{21}$$

where each $\mathfrak{g}_k$ *is a simple Lie algebra.*

Theorem 4 (Killing.) *The Lie algfebra* $\mathfrak{g}$ *is semisimple if and only if* $\det |g_{kp}| \neq 0$.

Example 3 $so(3)$ – $[X_k, X_l] = \epsilon_{klm} X_m$. *Thus for the adjoint representation we have* $(X_k)^T_{ml} = \epsilon_{klm}$, *i.e.:*

$$X_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \qquad X_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad X_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{22}$$

As a result from the definition (10) of the Killing form we get:

$$\begin{aligned} g_{11} &= \mathrm{tr}(X_1 \cdot X_1) = -2, \qquad g_{22} = \mathrm{tr}(X_2 \cdot X_2) = -2, \\ g_{33} &= \mathrm{tr}(X_3 \cdot X_3) = -2, \qquad g_{12} = g_{13} = g_{23} = 0. \end{aligned} \tag{23}$$

In another words $g_{kp} = -2\delta_{kp}$ for $so(3)$ and from theorem 4 it follows that $so(3)$ is semisimple or simple Lie algebra.

Example 4 *The Euclidean group in the plane $\mathbb{R}^2$ is given by:*

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x' \\ y' \end{pmatrix} = A(\theta) \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}, \quad A(\theta) = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}. \tag{24}$$

We can parametrize the group elements by:

$$A_E(\theta, a, b) = \begin{pmatrix} \cos(\theta) & \sin(\theta) & a \\ -\sin(\theta) & \cos(\theta) & b \\ 0 & 0 & 1 \end{pmatrix}, \tag{25}$$

so that (24) can be written in the form:

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = A(\theta, a, b) \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \tag{26}$$

The corresponding infinitesimal generators are:

$$X_\theta = \left.\frac{\partial A_E}{\partial \theta}\right|_{\theta=a=b=0} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad X_a = \left.\frac{\partial A_E}{\partial a}\right|_{\theta=a=b=0} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$X_b = \left.\frac{\partial A_E}{\partial b}\right|_{\theta=a=b=0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \tag{27}$$

and they obviously satisfy the commutation relations:

$$[X_\theta, X_a] = -X_b, \qquad [X_\theta, X_b] = X_a, \qquad [X_a, X_b] = 0. \tag{28}$$

$$X_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad X_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad X_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$
$$[X_1, X_2] = -X_3, \qquad [X_1, X_3] = X_2, \qquad [X_2, X_3] = 0. \tag{29}$$

The adjoint representation is given by

$$\mathrm{ad}_{X_1} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \equiv \begin{pmatrix} [X_1, X_1] \\ [X_1, X_2] \\ [X_1, X_3] \end{pmatrix} = \begin{pmatrix} 0 \\ -X_3 \\ X_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \tag{30}$$

Denoting by $\widetilde{X}_k = \mathrm{ad}_{X_k}$ *we have:*

$$\widetilde{X}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \widetilde{X}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \widetilde{X}_3 = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{31}$$

The metric tensor $g_{kl} = \mathrm{tr}\left(\widetilde{X}_k \widetilde{X}_l\right)$ *is equal to:*

$$g = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{32}$$

Check that $\widetilde{X}_k = -S X_k^t S$ *where* $S = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$*, i.e. in this case the adjoint representation is automorphic to the initial one.*

Example 5 $so(4)$ *has* 6 *generators which can be represented by the following* 4×4 *skew–*

symmetric matrices:

$$M_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad M_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$N_1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad N_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad N_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

which satisfy the commutation relations:

$$\begin{aligned} \left[\widehat{M}_i, \widehat{M}_j\right] &= \epsilon_{ijk}\widehat{M}_k, \qquad \left[\widehat{M}_i, \widehat{M}_i\right] = 0, \\ \left[\widehat{M}_i, \widehat{N}_j\right] &= \epsilon_{ijk}\widehat{N}_k, \qquad \left[\widehat{N}_i, \widehat{N}_j\right] = \epsilon_{ijk}\widehat{M}_k, \end{aligned} \tag{33}$$

It is instructive to change the basis to:

$$J_k = \frac{1}{2}\left(M_k + N_k\right), \qquad K_k = \frac{1}{2}\left(M_k - N_k\right), \tag{34}$$

$$\widehat{J}_k = \frac{1}{2}\left(\widehat{M_k} + \widehat{N}_k\right), \qquad \widehat{K}_k = \frac{1}{2}\left(\widehat{M_k} - \widehat{N}_k\right), \tag{35}$$

Then the commutation relations are simplified:

$$\left[\widehat{J}_i, \widehat{J}_j\right] = \epsilon_{ijk}\widehat{J}_k, \qquad \left[\widehat{J}_i, \widehat{K}_j\right] = 0, \qquad \left[\widehat{K}_i, \widehat{K}_j\right] = \epsilon_{ijk}\widehat{K}_k. \tag{36}$$

Thus the set of generators $\{\widehat{J}_1, \widehat{J}_2, \widehat{J}_3\}$ *and* $\{\widehat{K}_1, \widehat{K}_2, \widehat{K}_3\}$ *are separately closed under the commutation relations. Each of these sets satisfy the commutation relations of* $so(3)$*. So* $so(4)$ *may be considered as a direct sum of two different copies of* $so(3)$ *algebras:*

$$so(4) = so(3) \oplus so(3). \tag{37}$$

Each of the sets $\{\widehat{J}_1, \widehat{J}_2, \widehat{J}_3\}$ *and* $\{\widehat{K}_1, \widehat{K}_2, \widehat{K}_3\}$ *form a proper ideal in* $so(4)$*, so* $so(4)$ *is not simple but a semisimple Lie algebra.*

Definition 10 *One can also introduce a new linear operation in the algebra which formally may be called differentiation. This is a linear mapping:*

$$D : \mathfrak{g} \to \mathfrak{g}, \qquad D(\alpha X + \beta Y) = \alpha D(X) + \beta D(Y), \tag{38}$$

which satisfies:

$$D\left([X, Y]\right) = [D(X), Y] + [X, D(Y)]. \tag{39}$$

This resembles the formula for differentiation by parts with respect to the Lie–algebraic operation $[\ ,\ ]$. If we have two such operations D_1 and D_2, then $[D_1, D_2]$ is also a differentiation. Prove it by using the Jacobi identity.

An example of such differentiation is the ad_X operation, which by definition is:

$$\mathrm{ad}_X(Y) = [X, Y]\,, \tag{40}$$

where X is a fixed and Y is an arbitrary element of the algebra $\mathfrak{g}$. This operation is known also as the *adjoint action* of $\mathfrak{g}$ onto itself. One can prove, that:

$$\mathrm{ad}_{[X,Y]}(Z) = [\mathrm{ad}_X, \mathrm{ad}_Y]\,(Z), \tag{41}$$

which is again a consequence of the Jacobi identity.

Derivations of Lie Algebras. Let us introduce $\mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}]$ – the set of all possible commutators $[X, Y]$, where X and Y take any value in $\mathfrak{g}$. We can differentiate once again. If we define:

$$\mathfrak{g}^{(k)} = \left[\mathfrak{g}^{(k-1)}, \mathfrak{g}^{(k-1)}\right], \qquad k = 1, 2, \ldots \tag{42}$$

Then the sequence $\mathfrak{g}$, $\mathfrak{g}^{(1)}$, $\mathfrak{g}^{(2)}$, …, $\mathfrak{g}^{(k)}$, … is called the *derived series* of the Lie algebra $\mathfrak{g}$.

If for some positive integer k we have:

$$\mathfrak{g}^{(k)} = 0, \tag{43}$$

then $\mathfrak{g}$ is called a *solvable Lie algebra*.

Theorem 5 *If $\mathfrak{g}$ is a solvable Lie algebra then every Lie subalgebra of $\mathfrak{g}$ is also solvable.*

Starting from a Lie algebra $\mathfrak{g}$ we may also consider another sequence of algebras:

$$\mathfrak{g}^2 = \mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}], \qquad \mathfrak{g}^n = \left[\mathfrak{g}, \mathfrak{g}^{n-1}\right]. \tag{44}$$

The series

$$\mathfrak{g} \supseteq \mathfrak{g}^2 \supseteq \mathfrak{g}^3 \supseteq \cdots \tag{45}$$

is called the *descending central series* or descending sequence of ideals. If for some integer k

$$\mathfrak{g}^k = 0, \tag{46}$$

then $\mathfrak{g}$ is said to be *nilpotent*.

Exercise 1 *Show that $\mathfrak{g}^n$ are ideals of $\mathfrak{g}$.*

Theorem 6

1. *Every solvable Lie algebra can be represented by upper triangular matrices;*
2. *Every nilpotent Lie algebra can be represented by upper–triangular matrices with zeroes on the diagonal.*

Studying the structure of the ideals $\mathfrak{g}^1$, $\mathfrak{g}^2$, $\ldots$ one reaches the following conclusions:

- the series $\mathfrak{g}^1, \ldots, \mathfrak{g}^n, \ldots$ is always finite;
- after some large enough $n \geq N$ we have $\mathfrak{g}^{N-1} \supset \mathfrak{g}^N = \mathfrak{g}^{N+1} = \cdots$.

$\mathfrak{g}^N$ may have proper ideals that are not solvable. Then $\mathfrak{g}^N$ is a semisimple Lie algebra. If $\mathfrak{g}^N$ has no proper ideals then it is simple Lie algebra. In general each Lie algebra can be represented as a direct sum:

$$\mathfrak{g} = \mathbf{ss} \oplus \mathbf{r} \tag{47}$$

where ss means semisimple and $\mathbf{r}$ is a solvable radical.

More rigorously we shall say that $\mathfrak{g}$ is a direct sum of Lie subalgebras if:

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_n, \qquad \text{and} \qquad \mathfrak{g}_i \cap \mathfrak{g}_j = 0. \tag{48}$$

If $\mathfrak{g}$ has two subalgebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ such that

$$[\mathfrak{g}_1, \mathfrak{g}_1] \subset \mathfrak{g}_1, \qquad [\mathfrak{g}_1, \mathfrak{g}_2] \subset \mathfrak{g}_2, \qquad [\mathfrak{g}_2, \mathfrak{g}_2] \subset \mathfrak{g}_2, \tag{49}$$

then we shall say that $\mathfrak{g}$ is a semidirect sum:

$$\mathfrak{g} = \mathfrak{g}_1 \oplus_s \mathfrak{g}_2, \tag{50}$$

where $\mathfrak{g}_1$ is an ideal and $\mathfrak{g}_2$ is the residual part.

Theorem 7 *Any Lie algebra $\mathfrak{g}$ may be written as a semi–direct sum*

$$\mathfrak{g} = \mathcal{S} \oplus_s \mathcal{P}$$

where $\mathcal{P}$ is solvable and $\mathcal{S}$ is semisimple Lie algebra.

Classification of Lie Algebras

Structure of the Adjoint Representation

A Lie algebra $\mathfrak{g}$ is uniquely defined by its structure constants:

$$[X_i, X_j] = \sum_{k=1}^{r} {C_{ij}}^k X_k, \tag{51}$$

which must be skew–symmetric: ${C_{ij}}^k = -{C_{ji}}^k$
and satisfy Jacobi identity:

$$\sum_{m=1}^{L} \left({C_{ij}}^m {C_{mk}}^n + {C_{ki}}^m {C_{mj}}^n + {C_{jk}}^m {C_{mi}}^n\right) = 0.$$

With the help of ${C_{ij}}^m$ we also constructed the adjoint representation of $\mathfrak{g}$. If we think of X_k as a basic vector $|X_k\rangle$ in some r–dimensional real space then the adjoint action:

$$\mathrm{ad}_{X_i} \cdot X_k = [X_k, X_i] \tag{52}$$

can be rewritten as:

$$\mathrm{ad}_{X_i}|X_k\rangle = |[X_k, X_i]\rangle = -\sum_{m=1}^{r} C_{ik}{}^{m}\,|X_m\rangle \tag{53}$$

i.e.,

$$(\mathrm{ad}_{X_i})_k^m = -C_{ik}{}^{m}. \tag{54}$$

In this particular representation the algebraic operation $[X_k, X_i]$ is actually equivalent to matrix multiplication.

Obviously the structure constants contain excess information; so does the adjoint representation of $\mathfrak{g}$. Now we shall briefly discuss a way to cast the adjoint representation into a canonical form. We already introduced the notion of derived algebra and constructed the following sequence of ideals of $\mathfrak{g}$:

$$\mathfrak{g}, \quad \mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}], \quad \ldots, \quad \mathfrak{g}^{(k)} = [\mathfrak{g}, \mathfrak{g}^{(k-1)}]. \tag{55}$$

After a finite number of steps we reach to a subalgebra $\mathfrak{g}^{(N)}$ such that:

$$\mathfrak{g} \supset \mathfrak{g}^{(1)} \supset \cdots \supset \mathfrak{g}^{(N-1)} \supset \mathfrak{g}^{(N)} = \mathfrak{g}^{(N+1)} = \cdots \tag{56}$$

Let us now introduce a basis in $\mathfrak{g}$ in the following way:

$$\begin{array}{lll} X_k^{(1)} & \text{– span} & \mathfrak{g}/\mathfrak{g}^{(1)}, \qquad \text{(i.e.} \quad X_k^{(1)} \in \mathfrak{g} \quad \text{but} \quad X_k^{(1)} \notin \mathfrak{g}^{(1)};) \\ X_k^{(2)} & \text{– span} & \mathfrak{g}^{(1)}/\mathfrak{g}^{(2)} \\ X_k^{(N)} & \text{– span} & \mathfrak{g}^{(N-1)}/\mathfrak{g}^{(N)}, \\ X_k^{(N+1)} & \text{– span} & \mathfrak{g}^{(N)} \end{array} \tag{57}$$

Consider for simplicity $N = 2$ and let

$$X_k\text{– span} \quad \mathfrak{g}/\mathfrak{g}^{(1)}, \qquad Y_k\text{– span} \quad \mathfrak{g}^{(1)}/\mathfrak{g}^{(2)}, \tag{58}$$

and Z_k span $\mathfrak{g}^{(2)} = \mathfrak{g}^{(3)} = \cdots$. Every element in $\mathfrak{g}$ is a linear combination:

$$A = \sum_k A_k X_k + \sum_l A'_l Y_l + \sum_m A''_m Z_m. \tag{59}$$

Analogously if $B \in \mathfrak{g}^{(1)}$, then

$$B = \sum_l B'_l Y_l + \sum_m B''_m Z_m. \tag{60}$$

and if $C \in \mathfrak{g}^{(2)}$, then:

$$C = \sum_m C''_m Z_m. \tag{61}$$

Let us now calculate ad_X, $X \in \mathfrak{g}/\mathfrak{g}^{(1)}$. From the definition of $\mathfrak{g}^{(1)}$ we have:

$$\mathrm{ad}_X A \equiv [X, A] = b \in \mathfrak{g}^{(1)} \tag{62}$$

i.e.,

$$\mathrm{ad}_X A \equiv \sum_l b'_l Y_l + \sum_m b''_m Z_m. \tag{63}$$

This is true for all $A \in \mathfrak{g}$. However if $A = C \in \mathfrak{g}^{(2)}$ we can prove that the terms with Y_l will vanish. Indeed:

$$\mathrm{ad}_X A = \sum_m c''_m Z_m. \tag{64}$$

and by definition each Z_m can be written as:

$$Z_m = [Y'_m, Y''_m], \tag{65}$$

where Y'_m, Y''_m are conveniently choosen elements of $\mathfrak{g}^{(1)}$. Then using Jacobi identity we get:

$$\begin{aligned} [Z_m, X] &\equiv [[Y'_m, Y''_m], X] \\ &= -[[X, Y'_m,], Y''_m] - [[Y''_m, X], Y'_m] = 0. \end{aligned} \tag{66}$$

Again by definition $[X, Y'_m] \subset \mathfrak{g}^{(1)}$; $[Y''_m, X] \in \mathfrak{g}^{(1)}$ and Y'_m, $Y''_m \in \mathfrak{g}^{(1)}$. So both terms in the last equality belong to $\left[\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}\right] = \mathfrak{g}^{(2)}$. Thus we proved that ad_X has the following block–matrix structure:

$$\mathrm{ad}_X = \begin{array}{c} X \;\; Y \;\; Z \\ \begin{pmatrix} 0 & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \end{array} \begin{array}{l} X \in \mathfrak{g}/\mathfrak{g}^{(1)} \\ Y \in \mathfrak{g}^{(1)}/\mathfrak{g}^{(2)} \\ Z \in \mathfrak{g}^{(2)} \end{array} \tag{67}$$

where $*$ stays for some non–zero matrix elements. In analogous way one can deduce the matrix structure also of the other operators: ad_Y and ad_Z, $Y \in \mathfrak{g}^{(1)}/\mathfrak{g}^{(2)}$ and $Z \in \mathfrak{g}^{(2)}$. The final answer is that **all elements of $\mathfrak{g}$ in the adjoint representation have the same**

block–matrix structure:

$$\mathrm{ad}_A = \begin{pmatrix} 0 & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \quad \begin{matrix} \mathfrak{g}^{(0)}/\mathfrak{g}^{(1)} \\ \mathfrak{g}^{(1)}/\mathfrak{g}^{(2)} \\ \mathfrak{g}^{(2)} \end{matrix} \tag{68}$$

where $\mathfrak{g}^{(0)} \equiv \mathfrak{g}$. In the general case for $N > 2$ we get:

$$\mathrm{ad}_A = \begin{pmatrix} 0 & * & * & \dots & * & * \\ 0 & * & * & \dots & * & * \\ 0 & 0 & * & \dots & * & * \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & * & * \\ 0 & 0 & 0 & \dots & 0 & * \end{pmatrix} \quad \begin{matrix} \mathfrak{g}^{(0)}/\mathfrak{g}^{(1)} \\ \mathfrak{g}^{(1)}/\mathfrak{g}^{(2)} \\ \mathfrak{g}^{(2)}/\mathfrak{g}^{(3)} \\ \vdots \\ \mathfrak{g}^{(N-1)}/\mathfrak{g}^{(N)} \\ \mathfrak{g}^{(N)} \end{matrix} \tag{69}$$

Such structure is impossible only if $N = 1$, i.e. $\mathfrak{g} \equiv \mathfrak{g}^{(1)}$; then by definition $\mathfrak{g}$ is simple Lie algebra and only the right–hand down corner of ad_A is present.

Definition 11 *Suppose we have a matrix representation of $\mathfrak{g}$ (not necessarily the adjoint one) which has the block structure (69). Such representation we shall call reducible, but not fully reducible.*

Definition 12 *If the representation can be cast in a block–diagonal form:*

$$\mathrm{ad}_A = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & * & 0 & \dots & 0 & 0 \\ 0 & 0 & * & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & * & 0 \\ 0 & 0 & 0 & \dots & 0 & * \end{pmatrix} \tag{70}$$

then we shall call it fully reducible. If the representation of $\mathfrak{g}$ has no block–matrix structure then we call it irreducible.

Comparing these definitions with the definitions of solvable, semisimple and simple Lie algebras we get the following rule:

- $\mathfrak{g}$ is a *solvable* Lie algebra if its adjoint representation is *reducible, but not fully reducible*;

$$\mathfrak{g} = \begin{pmatrix} 0 & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \tag{71}$$

- $\mathfrak{g}$ is a *semisimple* Lie algebra if its adjoint representation is *fully reducible*;

$$\mathfrak{g} = \begin{pmatrix} * & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix} \tag{72}$$

- $\mathfrak{g}$ is a *simple* Lie algebra if its adjoint representation is *irreducible*.

$$\mathfrak{g} = (*)\,.$$

Bases in Classical Lie Groups and Algebras

We present here the well known parametrizations of the classical Lie algebras and some of their real forms.

Linear and Unitary Groups and Algebras.

$Gl(n, \mathbb{C})$

Any element M of this group can be represented as an $n \times n$ matrix:

$$X = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \dots & A_{nn} \end{pmatrix} = \sum_{i,j=1}^{n} A_{ij} E_{ij} \tag{73}$$

where the basic matrices E_{ij} are given by:

$$E_{ij} = \begin{array}{c} \qquad\qquad i \\ \begin{pmatrix} 0 & 0 & \dots & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 & \dots & 0 \end{pmatrix} \end{array} j \tag{74}$$

has just one non–zero matrix element different from zero and equal to one on the intersection of the i–th row and j–column. The matrix elements A_{ij} in (73) are complex numbers, i.e. $A_{ij} = a_{ij} + ib_{ij}$ where a_{ij} and b_{ij} are arbitrary real numbers.

$sl(n, \mathbb{C})$

same as $gl(n, \mathbb{C})$ with the restriction:

$$\mathrm{tr}X = \sum_{j=1}^{n} A_{jj} = 0; \tag{75}$$

$u(n, \mathbb{C})$

same as $gl(n, \mathbb{C})$ with the restriction:

$$X = -X^\dagger \qquad \text{i.e.} \qquad A_{ij} = -A_{ji}^* \tag{76}$$

which leaves as independent only the elements A_{ij} with $i \leq j$. The elements of the corresponding unitary group $U(n, \mathbb{C})$ are the exponentials $U = \exp(X)$. They satisfy:

$$UU^\dagger = \mathbb{1}. \tag{77}$$

and preserve the metric:

$$(\vec{z}, \vec{z}) \equiv \vec{z}^\dagger \vec{z} = \sum_{i=1}^{n} z_i^* z_i \tag{78}$$

$U(p, q, \mathbb{C})$ is another real form of the group $SL(n, \mathbb{C})$; its elements preserve the metric:

$$(\vec{z}, g\vec{z}) = \sum_{j,k=1}^{n} z_j^* g_{jk} z_k = \sum_{j=1}^{p} z_j^* z_j - \sum_{j=p+1}^{p+q} z_j^* z_j, \tag{79}$$

where

$$g_{jk} = \epsilon_j \delta_{jk}, \qquad \epsilon_j = \begin{cases} 1 & \text{for } j = 1, \ldots, p \\ -1 & \text{for } j = p+1, \ldots, p+q \end{cases} \tag{80}$$

The parametrization of $u(p, q, \mathbb{C})$ is obtained from the parametrization of $u(p+q, \mathbb{C})$ by the so called Weyl unitary trick:

$$\begin{pmatrix} X_{11} & X_{12} \\ -X_{12}^\dagger & X_{22} \end{pmatrix} \longrightarrow \begin{pmatrix} X_{11} & iX_{12} \\ -iX_{12}^\dagger & X_{22} \end{pmatrix} \tag{81}$$

where we have assumed a block–matrix structure; the diagonal blocks are skew–hermitian matrices:

$$X_{11} = -X_{11}^\dagger, \qquad X_{22} = -X_{22}^\dagger, \tag{82}$$

while X_{12} is an arbitrary complex $p \times q$ matrix.

$su(n, \mathbb{C})$

– same as $u(n, \mathbb{C})$ with tr $X = 0$;

$su(p, q, \mathbb{C})$

– same as $u(p, q, \mathbb{C})$ with tr X_{11} + tr $X_{22} = 0$;

$su^*(2n, \mathbb{C})$

The corresponding group $SU^*(2n)$ consists of all matrices in $SL(2n, \mathbb{C})$ which satisfy:

$$U^\dagger J U = J, \qquad \det U = 1, \tag{83}$$

with

$$J = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \tag{84}$$

Considering $U = \exp(i\alpha X)$ for small values of α we get that the elements of the corresponding Lie algebras $su^*(2n)$ must satisfy:

$$X^\dagger J - JX = 0, \qquad \text{tr}\, X = 0. \tag{85}$$

If we write down X as composed of 4 $n \times n$ blocks then (84) means that these blocks are interrelated by:

$$X = \begin{pmatrix} A & B \\ -B^* & A^* \end{pmatrix}, \qquad \text{tr}A + \text{tr}A^* = 0, \tag{86}$$

where A and B are arbitrary complex $n \times n$ matrices.

Orthogonal Groups and Algebras.

The orthogonal groups $so(n, \mathbb{R})$ consist of the transformations

$$\vec{x'} = U\vec{x}, \tag{87}$$

where $\vec{x} = (x_1, \ldots, x_n)^t \in \mathbb{R}^n$, which preserve the metric:

$$(\vec{x}, \vec{x}) = \sum_{k=1}^{n} x_k^2. \tag{88}$$

Obviously the requirement $(\vec{x'}, \vec{x'}) = (\vec{x}, \vec{x})$ means that the element $U \in SO(n)$ must satisfy:

$$U^t U = \mathbb{1} \tag{89}$$

$so(n,\mathbb{C})$ and $so(n,\mathbb{R})$

If $U = \exp(\alpha M)$ where α is a small parameter we get that the algebra $so(n,\mathbb{C})$ consist of the skew–symmetric matrices:

$$X = -X^t, \qquad \text{i.e.} \qquad A_{ij} = -A_{ji}, \tag{90}$$

where the matrix elements A_{ij} take arbitrary complex (for $so(n,\mathbb{C})$) or real (for $so(n,\mathbb{R})$) values respectively.

It will be natural to introduce as a basis in $so(n,\mathbb{C})$ the matrices:

$$\mathcal{O}_{ij} = E_{ij} - E_{ji}, \qquad i < j. \tag{91}$$

The group $so(n,\mathbb{R})$ allows an extension also to the complex case, or more exactly to the field of complex numbers. The elements $U \in so(n,\mathbb{C})$ will be considered as transformations (**??**) which preserve the bilinear form:

$$(\vec{z},\vec{z}) = \sum_{k=1}^{n} z_k^2. \tag{92}$$

Remark 1 *This is not a metric since $(\vec{z},\vec{z})$ may take arbitrary complex values. The conditions that must be satisfied by the corresponding Lie group and Lie algebra elements are like* *above, so $\mathcal{O}_{ij}$ form the basis also for $so(n, \mathbb{C})$. However now we consider X to be linear combination of $\mathcal{O}_{ij}$ with* complex–valued *coefficients.*

$so(p, q, \mathbb{R})$

consist of the transformations (87) which preserve the indefinite metric:

$$(\vec{x}, g\vec{x}) = \sum_{k,j=1}^{p+q} x_k g_{kj} x_j = \sum_{k=1}^{p} x_k^2 - \sum_{k=p+1}^{p+q} x_k^2, \tag{93}$$

where g is defined in (80). The element $U \in SO(p, q, \mathbb{R})$ must satisfy:

$$U^t g U = g, \tag{94}$$

and the elements of the corresponding Lie algebra $X \in so(p, q, \mathbb{R})$ satisfy:

$$X^t g + g X = 0, \tag{95}$$

i.e. they are g–skew–symmetric. Introducing X in block–matrix form in analogy with (81) we find that the general form of the elements of $so(p, q, \mathbb{R})$ can be obtained from the the one of

$so(p+q,\mathbb{C})$ again by the Weyl unitary trick (see (81)), which now has the form:

$$\begin{pmatrix} X_{11} & X_{12} \\ -X_{12}^t & X_{22} \end{pmatrix} \longrightarrow \begin{pmatrix} X_{11} & iX_{12} \\ -iX_{12}^t & X_{22} \end{pmatrix} \tag{96}$$

Here the diagonal blocks X_{11} and X_{22} are skew–symmetric:

$$X_{11} = -X_{11}^t, \qquad X_{22} = -X_{22}^t, \tag{97}$$

and X_{12} is arbitrary real $p \times q$ matrix.

The complex generalizations of these groups preserves the indefinite bilinear form:

$$(\vec{z}, g\vec{z}) = \sum_{k,j=1}^{p+q} z_k g_{kj} z_j = \sum_{k=1}^{p} z_k^2 - \sum_{k=p+1}^{p+q} z_k^2, \tag{98}$$

see the remark above concerning the complex generalization of $so(n)$. As basis in $so(p,q,\mathbb{C})$ and $so(p,q,\mathbb{R})$ we should use instead of (91):

$$\mathcal{O}_{ij} = E_{ij} - gE_{ji}g \tag{99}$$

$so^*(2n)$

The corresponding Lie group $SO^*(2n, \mathbb{C})$ is the subgroup of $SO(2n, \mathbb{C})$ which preserves the skew–symmetric metric:

$$\sum_{j=1}^{n} \left(z_k z^*_{n+k} - z_{n+k} z^*_k \right) . \tag{100}$$

The elements of the algebra $so^*(2n, \mathbb{C})$ have the form:

$$X = \begin{pmatrix} X_{11} & X_{12} \\ -X^*_{12} & X^*_{11} \end{pmatrix},$$
$$X_{11} = -X^t_{11}, \qquad X_{12} = X^\dagger_{12}. \tag{101}$$

We end this subsection by the commutation relations satisfied by the generators of $so(n)$:

$$[\mathcal{O}_{ij}, \mathcal{O}_{rs}] = \mathcal{O}_{is}\delta_{jr} + \mathcal{O}_{jr}\delta_{is} - \mathcal{O}_{ir}\delta_{js} - \mathcal{O}_{js}\delta_{ir}. \tag{102}$$