Cartan-Weyl bases and root systems of simple Lie algebras

Definition 1 Cartan subalgebra. *The Cartan subalgebra $\mathfrak{h} \in \mathfrak{g}$ is the maximal commutative subalgebra in $\mathfrak{g}$. Introduce a basis in $\mathfrak{h}$*

$$H_1, \dots, H_r, \qquad [H_j, H_k] = 0.$$

H_j – *Cartan generators of $\mathfrak{g}$.*

$\mathfrak{h}$ has the structure of Euclidean linear space.

Definition 2 Rank of $\mathfrak{g}$. $\dim \mathfrak{h} = r$ *is the rank of $\mathfrak{g}$.*

All elements of $\mathfrak{h}$ have common set of eigenvectors and eigensubspaces.

All elements of $\mathfrak{h}$ can be diagonalized simultaneously. This is specific property for $\mathfrak{g}$ over the complex numbers. For the real forms of $\mathfrak{g}$ this is not always possible.

From now on we assume that all elements of $\mathfrak{h}$ are represented as diagonal matrices.

Cartan-Weyl basis:

- Cartan generators:

$$H_k \in \mathfrak{h}, \qquad k = 1, \ldots r = \text{rank of } \mathfrak{g};$$

- Weyl generators E_α where $\alpha \in \Delta_{\mathfrak{g}}$ – the root system of $\mathfrak{g}$.

$$\text{ad}_H E_\alpha \equiv [H, E_\alpha] = (\alpha, \vec{h}) E_\alpha,$$

$$\text{ad}_H E_\alpha^T = -\left([H, E_\alpha]\right)^T = -(\alpha, \vec{h}) E_\alpha^T.$$

i.e. $E_\alpha^T \equiv E_{-\alpha}$ corresponds to the root $-\alpha$.

Classical series A_r: $\mathfrak{g} \simeq sl(n)$, $n = r + 1$.

$$
\begin{aligned}
H \in \mathfrak{h} \quad &\Leftrightarrow \quad \text{diag}\,(h_1, h_2, \ldots, h_r, h_{r+1}) \qquad n \times n \quad \text{diagonal matrix;} \\
H \in \mathfrak{h} \quad &\Leftrightarrow \quad \vec{h} = (h_1, h_2, \ldots, h_r, h_{r+1}) \qquad \in \mathbb{E}^n; \\
H \in \mathfrak{h} \quad &\Leftrightarrow \quad \text{ad}\,_H X = [H, X], \qquad \text{Linear operator in} \quad \mathfrak{g}.
\end{aligned}
\tag{1}
$$

where

$$\text{tr}\, H = \sum_{k=1}^{n} h_k = 0.$$

Let $\vec{e}_1, \vec{e}_2, \ldots, \vec{e}_n$ be orthonormal basis in $\mathbb{E}^n$:

$$(\vec{e}_i, \vec{e}_j) = \delta_{ij}.$$

Then $h_k = (\vec{h}, \vec{e}_k)$ and

$$(\vec{h}, \vec{\epsilon}) = 0, \qquad \vec{\epsilon} = \vec{e}_1 + \vec{e}_2 + \cdots + \vec{e}_n.$$

$\dim \mathfrak{h} = r$, i.e. rank $sl(r+1) = r$.

Weyl generators: they do not commute with $\mathfrak{h}$, i.e. they are not diagonal. They are classified as eigenvectors of ad $_H\cdot$. Formal definition:

$$\text{ad}\,_H E_\alpha \equiv [H, E_\alpha] = \alpha(H)E_\alpha.$$

the root α – linear functional on $\mathfrak{h}$; E_α is the corresponding root vector.

$$E_\alpha \equiv E_{ij}, \qquad (E_{ij})_{pq} = \delta_{ip}\delta_{jq}.$$

Start with:

$$\text{ad}\,_H E_{ij} \equiv [H, E_{ij}] = (h_i - h_j)E_{ij} = (\vec{h}, \vec{e}_i - \vec{e}_j)E_{ij},$$

$$[H, E_\alpha] = (\vec{h}, \alpha)E_\alpha = \alpha(H)E_\alpha.$$

where

$$\alpha = \vec{e}_i - \vec{e}_j.$$

The vector $\alpha \in \mathbb{E}^n$ is one of the roots of $sl(n)$. The set of all roots is

$$\Delta_{A_r} \equiv \{\vec{e}_i - \vec{e}_j, \quad 1 \le i \ne j \le r+1\}$$

The set of positive roots are:

$$\Delta^+_{A_r} \equiv \{\vec{e}_i - \vec{e}_j, \quad 1 \le i < j \le r+1\}$$

Example: $r = 1$, $\mathfrak{g} \simeq sl(2)$. The root system consists of 2 roots orthogonal to $\vec{e}_1 + \vec{e}_2$

$$\alpha = (\vec{e}_1 - \vec{e}_2), \qquad -\alpha = -(\vec{e}_1 - \vec{e}_2).$$

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad E_\alpha = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad E_{-\alpha} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

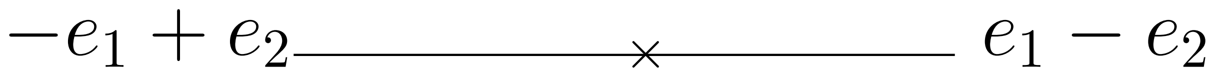

Figure 1: The root system of $A_1 \simeq sl(2)$.

Example: $r = 2$**,** $\mathfrak{g} \simeq sl(3)$**.** The root system consists of 6 roots orthogonal to $\vec{e}_1 + \vec{e}_2 + \vec{e}_3$

$$\pm(\vec{e}_1 - \vec{e}_2), \quad \pm(\vec{e}_1 - \vec{e}_3), \quad \pm(\vec{e}_2 - \vec{e}_3).$$

Positive roots:

$$(\vec{e}_1 - \vec{e}_2), \quad (\vec{e}_1 - \vec{e}_3), \quad (\vec{e}_2 - \vec{e}_3).$$

Cartan-Weyl basis of $sl(3)$:

$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

$$E_{e_1-e_2} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_{e_2-e_3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_{e_1-e_3} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$E_{-e_1+e_2} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_{-e_2+e_3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad E_{-e_1+e_3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

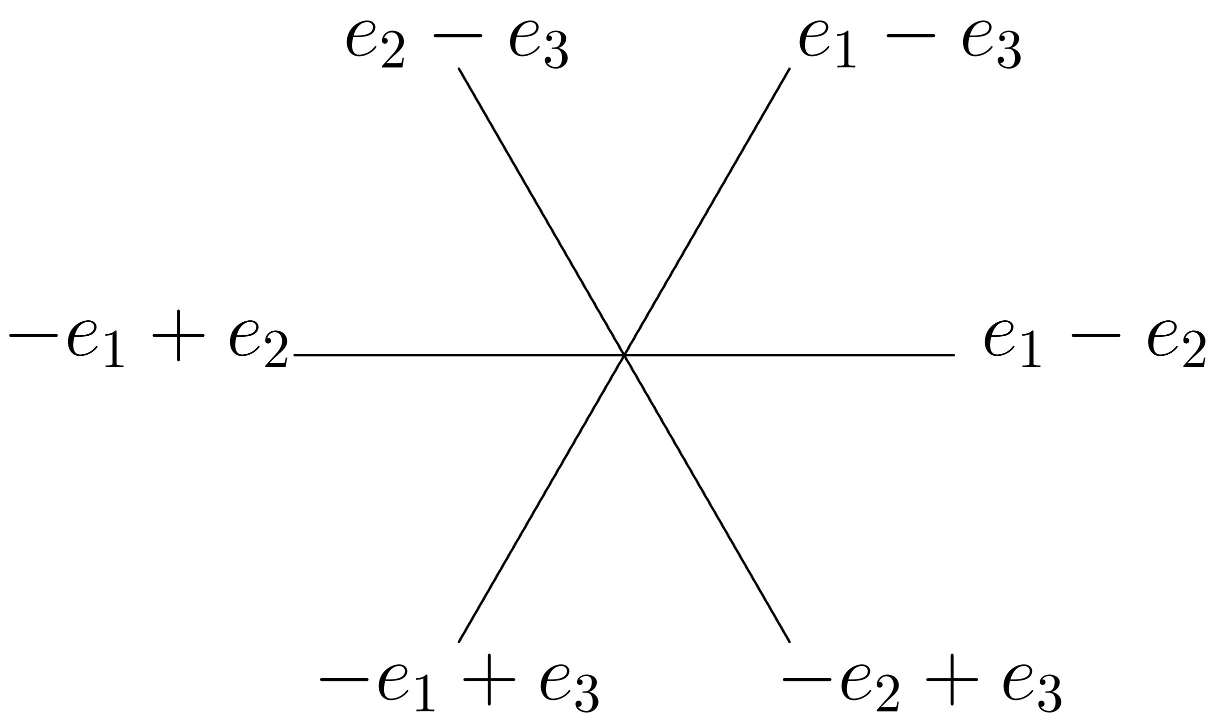

Figure 2: The root system of $A_2 \simeq sl(3)$.

B_r-series – $so(2r+1)$-algebras.

Need a redefinition of $\mathfrak{g}$ in order that the Cartan subalgebra is diagonal.

$$X \in so(2r) \leftrightarrow XS_0 + S_0X^T = 0, \quad \text{tr}\, X \equiv \sum_{k=1}^{2r} X_{kk} = 0.$$

For convenience we use S_0

$$S_0 = \left(\begin{array}{ccc|c|ccc} 0 & 0 & \dots & \dots & \dots & 0 & 1 \\ 0 & 0 & \dots & \dots & \dots & -1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ \hline \vdots & \vdots & \ddots & (-1)^r & \vdots & \vdots & \vdots \\ \hline \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & -1 & \dots & \dots & \dots & 0 & 0 \\ 1 & 0 & \dots & \dots & \dots & 0 & 0 \end{array}\right) = \sum_{k=1}^{r} (-1)^{2r+2-k} (E_{k\bar{k}} + E_{\bar{k}k}) + (-1)^r E_{r+1,r+1},$$

where $\bar{k} = 2r + 2 - k$. Important property: $S_0^2 = \mathbb{1}$.

$$H \in \mathfrak{h} \quad \Leftrightarrow \quad \text{diag}\,(h_1, h_2, \ldots, h_r, 0, -h_r, \cdots - h_2, -h_1) \quad 2r+1 \times 2r+1 \text{ diagonal matr}$$

$$H \in \mathfrak{h} \quad \Leftrightarrow \quad \vec{h} = (h_1, h_2, \ldots, h_r) \quad \in \mathbb{E}^r;$$

$$H \in \mathfrak{h} \quad \Leftrightarrow \quad \text{ad}\,_H X = [H, X], \quad \text{Linear operator in } \mathfrak{g}.$$

$\dim \mathfrak{h} = r$, i.e. rank $so(2r+1) = r$.

Duality between $\mathfrak{h}$ and r-dimensional Euclidean space $\mathbb{E}^r$.

Weyl generators: they are the rest of the generators, that do not commute with $\mathfrak{h}$, i.e. they are not diagonal. They are classified as eigenvectors of ad $_H \cdot$.

$$\text{ad}\,_H E_\alpha = \alpha(H) E_\alpha = (\alpha, \vec{h}) E_\alpha.$$

the root α – linear functional on $\mathfrak{h}$;

E_α is the corresponding root vector.

What are the eigenvalues of ad $_H$? Let $r = 2$, $\mathfrak{g} \simeq so(5)$.

$$S_0 = \left(\begin{array}{cc|c|cc} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \\ \hline 0 & 0 & 1 & 0 & 0 \\ \hline 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{array}\right), \qquad S_0 X + X^T S_0 = 0, \qquad S_0^2 = 1\!\!1.$$

$$H = \left(\begin{array}{cc|c|cc} h_1 & 0 & 0 & 0 & 0 \\ 0 & h_2 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & -h_2 & 0 \\ 0 & 0 & 0 & 0 & -h_1 \end{array}\right), \qquad \dim \mathfrak{h} = 2, \qquad \text{rank}\, so(5) = 2.$$

$$E_\alpha \leftrightarrow \text{ad}\,_H E_\alpha \equiv [H, E_\alpha] = (\alpha, \vec{h}) E_\alpha. \qquad E_\alpha = \left(\begin{array}{cc|c|cc} 0 & * & + & \circ & 0 \\ 0 & 0 & \# & 0 & \circ \\ \hline 0 & 0 & 0 & \# & + \\ \hline 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

Four possibilities $h_i \pm h_j = (\vec{h}, e_i \pm e_j)$:

$$[H, E_\alpha] = (h_1 - h_2) E_\alpha, \qquad E_\alpha = \left(\begin{array}{cc|c|cc} 0 & * & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 \end{array} \right), \qquad \alpha = \vec{e}_1 - \vec{e}_2,$$

$$[H, E_\alpha] = (h_1 + h_2) E_\alpha, \qquad E_\alpha = \left(\begin{array}{cc|c|cc} 0 & 0 & 0 & \circ & 0 \\ 0 & 0 & 0 & 0 & \circ \\ \hline 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right), \qquad \alpha = \vec{e}_1 + \vec{e}_2,$$

$$[H, E_\alpha] = h_1 E_\alpha, \qquad E_\alpha = \left(\begin{array}{cc|c|cc} 0 & 0 & + & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & + \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right), \qquad \alpha = \vec{e}_1,$$

$$[H, E_\alpha] = h_2 E_\alpha, \qquad E_\alpha = \left(\begin{array}{cc|c|cc} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \# & 0 & 0 \\ \hline 0 & 0 & 0 & \# & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right), \qquad \alpha = \vec{e}_2.$$

These roots are called positive, because E_α are upper triangular matrices. There are four more possibilities corresponding to negative roots $-\alpha$ with $E_{-\alpha} = E_\alpha^T$.

In general

$$\Delta_{B_r} \equiv \{\pm(\vec{e}_i - \vec{e}_j), \quad \pm(\vec{e}_i + \vec{e}_j), \quad i < j, \qquad \pm\vec{e}_j\}$$

$$\Delta_{B_r}^+ \equiv \{(\vec{e}_i - \vec{e}_j), \quad (\vec{e}_i + \vec{e}_j), \quad i < j, \qquad \vec{e}_j\}.$$

Short and long roots are present.

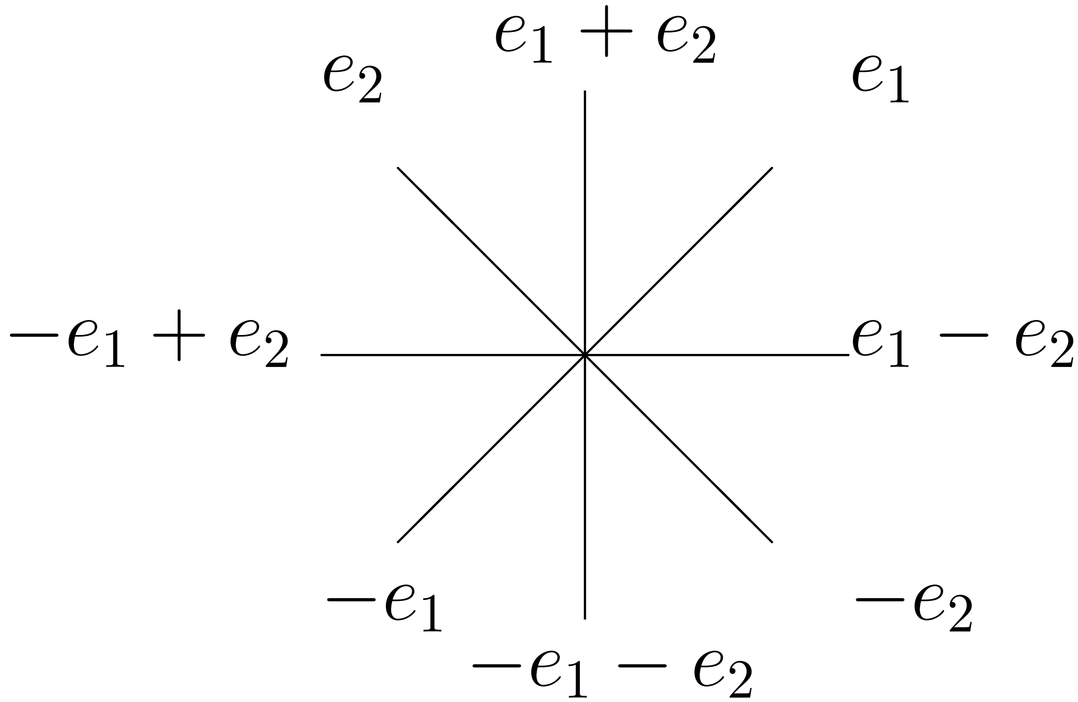

Figure 3: The root system of the $B_2 \simeq so(5)$.

$sp(2r)$-algebras – C_r-series.

$$X \in sp(2r) \leftrightarrow XS_1 + S_1 X^T = 0, \quad \text{tr}\, X \equiv \sum_{k=1}^{2r} X_{kk} = 0.$$

For convenience we slightly change S_1. Instead of

$$S_1 = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}, \qquad S_1^2 = -\mathbb{1}.$$

we now take

$$S_1 = \left(\begin{array}{ccc|ccc} 0 & 0 & \dots & \dots & 0 & 1 \\ 0 & 0 & \dots & \dots & -1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \hline \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & \dots & \dots & 0 & 0 \\ -1 & 0 & \dots & \dots & 0 & 0 \end{array}\right) = \sum_{k=1}^{r} (-1)^{2r+1-k} (E_{k\bar{k}} + E_{\bar{k}k}), \qquad \bar{k} = 2r + 1 - k$$

Important property: $S_1^2 = -\mathbb{1}$ is preserved.

$$
\begin{array}{llll}
H \in \mathfrak{h} & \Leftrightarrow & \text{diag}\,(h_1, h_2, \ldots, h_r, -h_r, \cdots - h_2, -h_1) & 2r \times 2r \quad \text{diagonal matrix;} \\
H \in \mathfrak{h} & \Leftrightarrow & \vec{h} = (h_1, h_2, \ldots, h_r) & \in \mathbb{E}^r; \\
H \in \mathfrak{h} & \Leftrightarrow & \text{ad}\,_H X = [H, X], & \text{Linear operator in} \quad \mathfrak{g}.
\end{array} \tag{2}
$$

$\dim \mathfrak{h} = r$, i.e. rank $sp(2r) = r$.

Weyl generators: they are the rest of the generators, that do not commute with $\mathfrak{h}$, i.e. they are not diagonal. They are classified as eigenvectors of ad $_H\cdot$. Formal definition:

$$
\text{ad}\,_H E_\alpha \equiv [H, E_\alpha] = \alpha(H) E_\alpha = (\alpha, \vec{h}) E_\alpha .
$$

the root α – linear functional on $\mathfrak{h}$;
E_α is the corresponding root vector.

What are the eigenvalues of ad $_H$? Let $r = 2$, $\mathfrak{g} \simeq sp(4)$.

$$S_1 = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{array}\right), \qquad S_1 X + X^T S_1 = 0, \qquad S_1^2 = -\mathbb{1}.$$

$$H = \left(\begin{array}{cc|cc} h_1 & 0 & 0 & 0 \\ 0 & h_2 & 0 & 0 \\ \hline 0 & 0 & -h_2 & 0 \\ 0 & 0 & 0 & -h_1 \end{array}\right), \qquad \dim \mathfrak{h} = 2, \qquad \text{rank}\, sp(4) = 2.$$

$$E_\alpha \leftrightarrow \text{ad}_H E_\alpha \equiv [H, E_\alpha] = (\alpha, \vec{h}) E_\alpha, \qquad E_\alpha = \left(\begin{array}{cc|cc} 0 & * & + & \circ \\ 0 & 0 & \# & + \\ \hline 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{array}\right),$$

Four possibilities $h_i \pm h_j = (\vec{h}, e_i \pm e_j)$:

$$[H, E_\alpha] = (h_1 - h_2) E_\alpha, \qquad E_\alpha = \left(\begin{array}{cc|cc} 0 & * & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{array}\right), \qquad \alpha = \vec{e}_1 - \vec{e}_2,$$

$$[H, E_\alpha] = (h_1 + h_2) E_\alpha, \qquad E_\alpha = \left(\begin{array}{cc|cc} 0 & 0 & + & 0 \\ 0 & 0 & 0 & + \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right), \qquad \alpha = \vec{e}_1 + \vec{e}_2,$$

$$[H, E_\alpha] = 2 h_1 E_\alpha, \qquad E_\alpha = \left(\begin{array}{cc|cc} 0 & 0 & 0 & \circ \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right), \qquad \alpha = 2\vec{e}_1,$$

$$[H, E_\alpha] = 2h_2 E_\alpha, \qquad E_\alpha = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 0 \\ 0 & 0 & \circ & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right), \qquad \alpha = 2\vec{e}_2.$$

These roots are called positive, because E_α are upper triangular matrices. There are 4 more possibilities corresponding to negative roots $-\alpha$ with $E_{-\alpha} = E_\alpha^T$.

In general

$$\Delta_{C_r} \equiv \{\pm(\vec{e}_i - \vec{e}_j), \quad \pm(\vec{e}_i + \vec{e}_j), \quad i < j, \quad \pm 2\vec{e}_i, \}$$

$$\Delta_{C_r}^+ \equiv \{(\vec{e}_i - \vec{e}_j), \quad (\vec{e}_i + \vec{e}_j), \quad i < j, \quad 2\vec{e}_i, \}$$

Short and long roots are present.

Remark 1 *The root systems of B_2 and C_2 are equivalent. No equivalence between B_r and C_r for $r \geq 3$.*

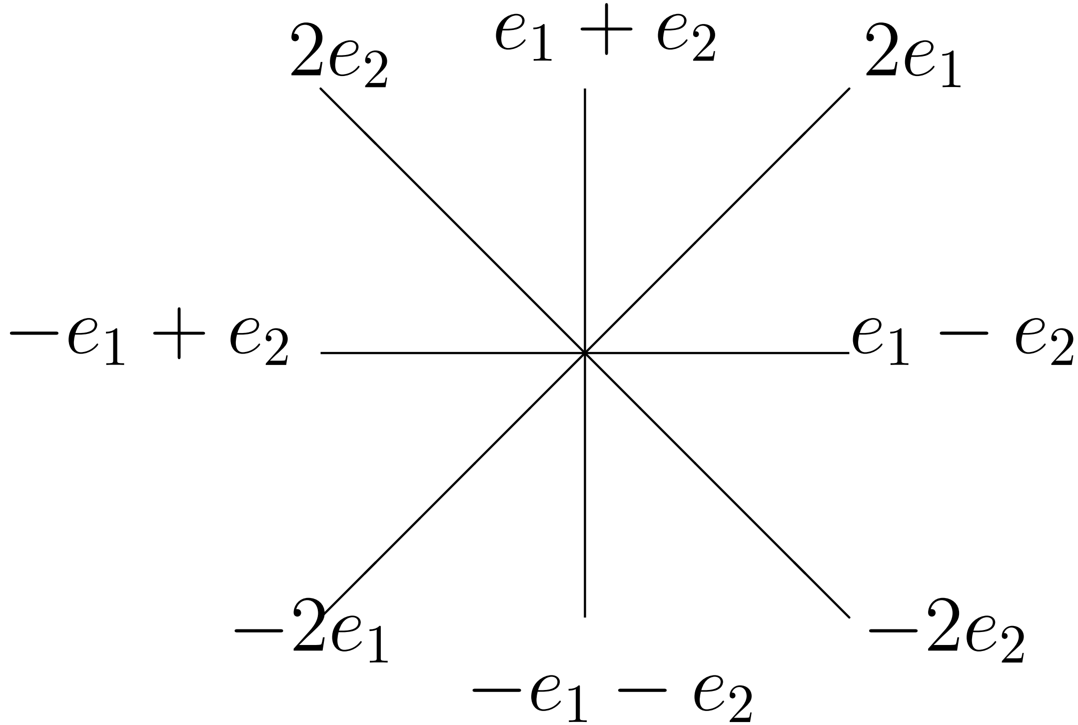

Figure 4: The root system of the $C_2 \simeq sp(4)$.

$so(2r)$-algebras – D_r-series.

Need a redefinition of $\mathfrak{g}$ in order that the Cartan subalgebra is diagonal.

$$X \in so(2r) \leftrightarrow XS_0 + S_0 X^T = 0, \quad \mathrm{tr}\, X \equiv \sum_{k=1}^{2r} X_{kk} = 0.$$

For convenience we use S_0

$$S_0 = \left(\begin{array}{ccc|ccc} 0 & 0 & \dots & \dots & 0 & 1 \\ 0 & 0 & \dots & \dots & -1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \\ \hline \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & -1 & \dots & \dots & 0 & 0 \\ 1 & 0 & \dots & \dots & 0 & 0 \end{array}\right) = \sum_{k=1}^{r} (-1)^{2r+1-k} (E_{k\bar{k}} - E_{\bar{k}k}), \qquad \bar{k} = 2r + 1 - k$$

Important property: $S_0^2 = \mathbb{1}$.

$$
\begin{array}{llll}
H \in \mathfrak{h} & \Leftrightarrow & \text{diag}\,(h_1, h_2, \dots, h_r, -h_r, \dots - h_2, -h_1) & 2r \times 2r \text{ diagonal matrix;} \\
H \in \mathfrak{h} & \Leftrightarrow & \vec{h} = (h_1, h_2, \dots, h_r) & \in \mathbb{E}^r; \\
H \in \mathfrak{h} & \Leftrightarrow & \text{ad}_H X = [H, X], & \text{Linear operator in } \mathfrak{g}.
\end{array} \tag{3}
$$

$\dim \mathfrak{h} = r$, i.e. rank $so(2r) = r$.

Weyl generators: they are the rest of the generators, that do not commute with $\mathfrak{h}$, i.e. they are not diagonal. They are classified as eigenvectors of ad $_H \cdot$.

$$\text{ad}_H E_\alpha \equiv [H, E_\alpha] = \alpha(H) E_\alpha = (\alpha, \vec{h}) E_\alpha.$$

the root α – linear functional on $\mathfrak{h}$;
E_α is the corresponding root vector.

What are the eigenvalues of ad $_H$? Let $r = 2$, $\mathfrak{g} \simeq so(4)$.

$$S_0 = \left(\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ \hline 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right), \qquad S_0 X + X^T S_0 = 0,$$

$$H = \left(\begin{array}{cc|cc} h_1 & 0 & 0 & 0 \\ 0 & h_2 & 0 & 0 \\ \hline 0 & 0 & -h_2 & 0 \\ 0 & 0 & 0 & -h_1 \end{array}\right), \qquad \dim \mathfrak{h} = 2, \qquad \text{rank}\, so(4) = 2.$$

$$E_\alpha \leftrightarrow \text{ad}_H E_\alpha \equiv [H, E_\alpha] = (\alpha, \vec{h}) E_\alpha. \qquad E_\alpha = \left(\begin{array}{cc|cc} 0 & * & + & 0 \\ 0 & 0 & 0 & + \\ \hline 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{array}\right),$$

Two possibilities:

$$[H, E_\alpha] = (h_1 - h_2) E_\alpha, \qquad E_\alpha = \left(\begin{array}{cc|cc} 0 & * & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 \end{array}\right), \qquad \alpha = \vec{e}_1 - \vec{e}_2,$$

$$[H, E_\alpha] = (h_1 + h_2) E_\alpha, \qquad E_\alpha = \left(\begin{array}{cc|cc} 0 & 0 & + & 0 \\ 0 & 0 & 0 & + \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right), \qquad \alpha = \vec{e}_1 + \vec{e}_2,$$

These roots are called positive, because E_α are upper triangular matrices. There are two more possibilities corresponding to negative roots $-\alpha$ with $E_{-\alpha} = E_\alpha^T$. In general

$$\Delta_{D_r} \equiv \{\pm(\vec{e}_i - \vec{e}_j), \quad \pm(\vec{e}_i + \vec{e}_j), \quad i < j, \}$$

$$\Delta_{D_r}^+ \equiv \{(\vec{e}_i - \vec{e}_j), \quad (\vec{e}_i + \vec{e}_j), \quad i < j, \}$$

All roots are of the same length.

Example: $r = 2$**,** $\mathfrak{g} \simeq so(4)$**.** The root system consists of 4 roots

$$\pm(\vec{e}_1 - \vec{e}_2), \qquad \pm(\vec{e}_1 + \vec{e}_2).$$

$$H_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad H_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$E_{e_1 - e_2} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad E_{e_1 + e_2} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad E_{-\alpha} = E_\alpha^T.$$

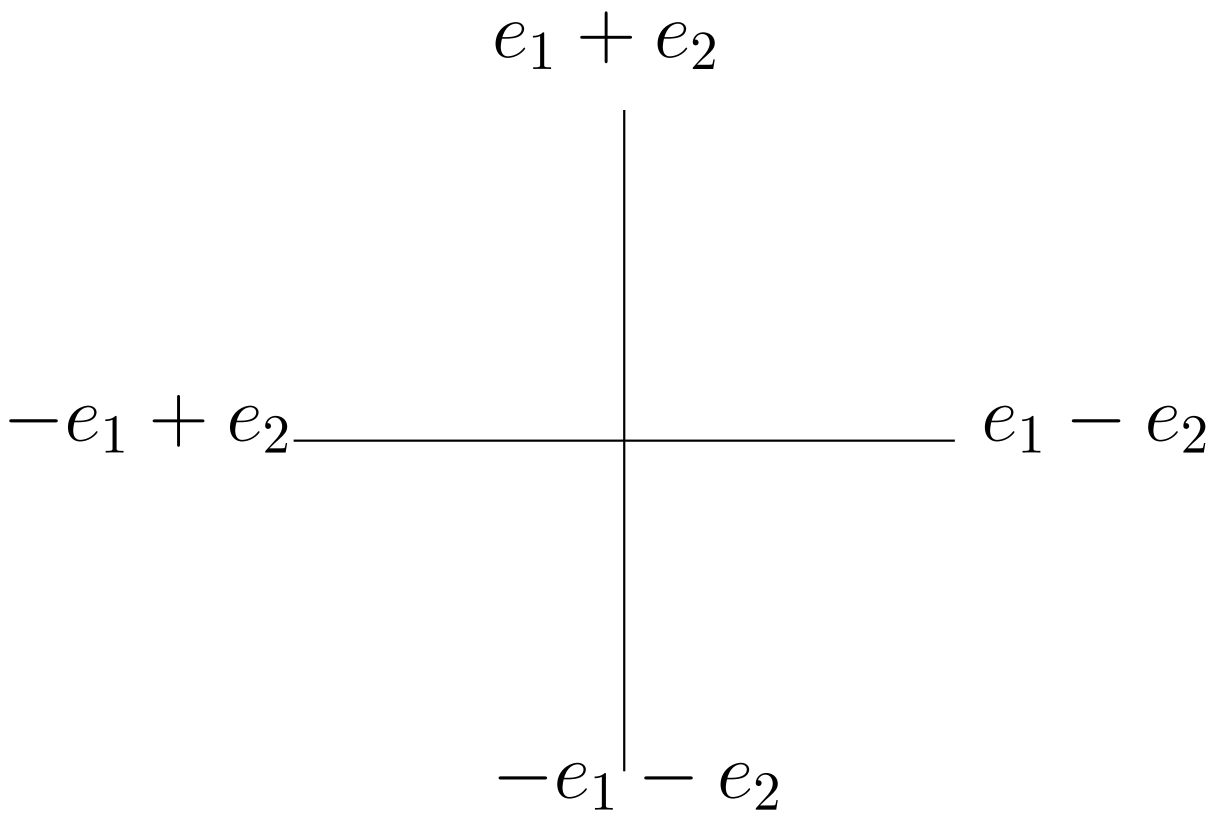

Figure 5: The root system of $D_2 = A_1 \oplus A_1$; i.e. $so(4) \equiv sl(2) \oplus sl(2)$.

Remark 2 *i) The root system of* $D_2 \simeq so(4)$ *consists of two pairs of roots that are mutually orthogonal. So* $so(4) = so(2) \oplus so(2)$*;*
ii) The root system of $D_3 \simeq so(6)$ *is equivalent to the root system of* $A_3 \simeq sl(4)$*. So* $so(6) \simeq sl(4)$*.*

The Cartan-Weyl basis

It consists of Cartan generators $H_k \in \mathfrak{h}$.

Important: to each element $H \in \mathfrak{h}$ there corresponds a vector in the root space $\mathbb{E}^r$. For example we can choose H_k so that they are dual to $\vec{e}_k \in \mathbb{E}^r$.

There are also Weyl generators E_α:

$$\text{ad}_H E_\alpha = \alpha(H) E_\alpha, \qquad \alpha \in \Delta$$

α are the roots, the set of all roots form the root system Δ of $\mathfrak{g}$.

$$\mathfrak{g} = \oplus \mathfrak{h} \bigoplus_{\alpha \in \Delta} \mathfrak{g}^\alpha.$$

Theorem. $\dim \mathfrak{g}^\alpha = 1$ for all $\alpha \in \Delta$.

Remark: This is valid only for simple Lie algebras over $\mathbb{C}$.

Theorem. The Killing form $B(X, Y)$ is non-degenerate on the Cartan subalgebra $\mathfrak{h}$.

$$B(E_\alpha, E_\beta) = 0, \qquad \text{if} \quad \alpha + \beta \neq 0, \qquad B(E_\alpha, E_{-\alpha}) = 1,$$

if E_α is properly normalized.

We need the commutation relations for the Cartan-Weyl basis:

$$[E_\alpha, E_\beta] = N_{\alpha,\beta} E_{\alpha+\beta}, \qquad \text{if} \qquad 0 \neq \alpha + \beta \in \Delta.$$

$$[E_\alpha, E_{-\alpha}] = H_\alpha$$

where $H_\alpha \in \mathfrak{h}$ is the element dual to the root α.

Proof. Let $H \in \mathfrak{h}$. Then

$$\begin{aligned}
&[H, [E_\alpha, E_\beta]] = -[E_\alpha, [E_\beta, H]] - [E_\beta, [H, E_\alpha]] = \\
&= \beta(H)[E_\alpha, E_\beta] - \alpha(H)[E_\beta, E_\alpha] = \\
&= (\alpha(H) + \beta(H))[E_\alpha, E_\beta]
\end{aligned}$$

Therefore

$$[\mathfrak{g}^\alpha, \mathfrak{g}^\beta] \in \mathfrak{g}^{\alpha+\beta} \quad \text{if} \quad \alpha + \beta \in \Delta, \qquad [\mathfrak{g}^\alpha, \mathfrak{g}^\beta] = 0 \quad \text{if} \quad \alpha + \beta \notin \Delta.$$

But since $\dim \mathfrak{g} = 1$ this proves the first relation.

For the second relation we use the duality between $\mathfrak{h}$ and the root space $\mathbb{E}^r$. Let H and H' be dual to the vectors $\vec{h} \in \mathbb{E}^r$ and $\vec{h}' \in \mathbb{E}^r$ respectively. The Killing form $B(H, H')$ provides metric on $\mathfrak{h}$ which corresponds to the metric in $\mathbb{E}^r$. Then it is possible to identify

$$B(H,H') = (\vec{h},\vec{h}'),$$

$$\begin{aligned} &B(H,[E_\alpha,E_{-\alpha}]) = B([H,E_\alpha],E_{-\alpha}) \\ &= \alpha(H)B(E_\alpha,E_{-\alpha}) = \alpha(H) = (\alpha,\vec{h}) \neq 0. \end{aligned}$$

which must hold true for any H (respectively, for any $\vec{h}$). The only way this could hold true is to choose

$$\vec{h}' = \alpha \leftrightarrow H' = H_\alpha,$$

i.e.

$$[E_\alpha,E_{-\alpha}] = H_\alpha \in \mathfrak{h}.$$

Finally we have derived the following commutation relations between the elements of the Cartan–Weyl basis:

$$\begin{aligned} [H,E_{-\alpha}] &= \alpha(H)E_\alpha = (\alpha,\vec{h})E_\alpha, \\ [E_\alpha,E_{-\alpha}] &= H_\alpha, \\ [E_\alpha,E_\beta] &= N_{\alpha,\beta}E_{\alpha+\beta} \qquad \text{for} \quad \alpha+\beta \in \Delta. \end{aligned}$$

From now on we shall suppose that the normalization condition $B(E_\alpha, E_{-\alpha}) = 1$ holds.

Properties of $N_{\alpha,\beta}$

Lemma

$$\begin{array}{lll} \text{a)} & N_{\alpha,\beta} = -N_{\beta,\alpha} & \text{skew-symmetricity;} \\ \text{b)} & N_{\alpha,\beta} = 0 \quad \text{if} \quad \alpha+\beta \notin \Delta; & \\ \text{c)} & N_{\alpha,\beta} = N_{\beta,\gamma} = N_{\gamma,\alpha} & \end{array}$$

for all α, β, $\gamma \in \Delta$ and such that $\alpha + \beta + \gamma = 0$.

Properties of the root systems

We shall list the main properties of the root systems Δ.

According to Coxeter theory one can put them as axioms and try to describe all sets of vectors in the finite dimensional Euclidean space $\mathbb{E}^l$ which satisfy these axioms. The main result of Coxeter consist in the proof that all such sets of vectors in fact are given by the systems of roots of the semisimple Lie algebras.

1. Δ is a finite set, generating E^l;

2. if $\alpha \in \Delta$ then $2\alpha \notin \Delta$;

3. $\dfrac{2(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$ for all $\alpha, \beta \in \Delta$;

4. $S_\alpha \Delta \equiv \Delta$, where $S_\alpha \vec{x} = \vec{x} - \dfrac{2(\alpha,\beta)}{(\alpha,\alpha)}\alpha$ is the reflection with respect to the plane, orthogonal to the root α.

5. **Consequence:** if $\alpha \in \Delta$, then $-\alpha \in \Delta$. Indeed, apply S_α to $\vec{x} = \alpha$.

The root system of $sl(2)$ is embedded in one dimensional space E^1 and consists of only two vectors: $\{\alpha, -\alpha\}$. The Cartan–Weyl basis in the fundamental two–dimensional representation is given by:

$$\dim \mathfrak{h} = 1, \qquad H_1 = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{4}$$

$$E_\alpha = \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad E_{-\alpha} = \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

Now we shall try to answer the question: what is the *minimal information* which uniquely determines given simple Lie algebra?

Obviously $\mathfrak{g}$ is uniquely determined by its structure constants C_{kl}^{m}, but these are not independent. They must satisfy $C_{kl}^{m} = -C_{lk}^{m}$ and also the Jacobi identity.

The introduction of $\mathfrak{h}$ and Δ explicitly solved part of the resytictions on C_{kl}^{m}. Indeed, knowing the root system Δ we can determine the commutation relations between the elements of the Cartan-Weyl basis. An important role here is played by the lemma allowing one to express $\tilde{N}_{\alpha,\beta}$ by the length of the α-series of roots containing β: $\tilde{N}_{\alpha,\beta} = \pm(1-p)$.

However, Δ also contains superfluous information, which is a consequence of the high symmetry of Δ:

$$S_\alpha \Delta = \Delta, \qquad \forall \alpha \in \Delta. \tag{5}$$

In order to answer the question, asked above we introduce two notions: the notion of ordering in Δ, and the notion of a set of *simple roots* of $\mathfrak{g}$.

We shall say that the *root* $\alpha \in \Delta$ *is positive (negative)* if its first nonzero component $(\alpha^{(1)}, \alpha^{(2)}, \ldots$ is *positive (negative)*.

A system of roots $(\alpha_1, \ldots, \alpha_r)$ is called a *system of simple roots of* $\mathfrak{g}$ if:

$$\text{a)} \qquad \alpha_j > 0 \qquad \text{linearly independent;} \tag{6}$$

$$\text{b)} \qquad \alpha_i - \alpha_j \qquad \text{is \textit{not} a root for all} \quad i \neq j. \tag{7}$$

Theorem i) Every root $\alpha \in \Delta$ can be expressed as linear combination of the simple ones:

$$\alpha = \sum_{k=1}^{r} \alpha_k m_k \tag{8}$$

where either *all* $m_k \geq 0$ and integer and then $\alpha > 0$; or *all* $m_k \leq 0$ and integer and then $\alpha < 0$;
ii) Each root can be obtained by a reflection $S_\gamma \in W_{\mathfrak{g}}$ from a conveniently chosen simple root.
Definitions.

- $\sum_{k=1}^{r} = \mathrm{ht}\,(\alpha)$ is called the height of the root α.
- The root $\alpha_{\max}$ is maximal if $\alpha_{\max} + \alpha \notin \Delta$ for any positive root $\alpha > 0$.
- $C = S_{\alpha_1} \ldots S_{\alpha_r}$ is known as the Coxeter element in the Weyl group. The number h for which $C^h = \mathbb{1}$ is known as the Coxeter number of $\mathfrak{g}$.

Theorem 1. The Coxeter number h does not depend on: i) the choice of the Cartan-Weyl basis; ii) on the choice of the set of simple roots; iii) on the order in which the reflections S_{α_k} are taken.
2. ht $(\alpha_{\max}) = h - 1$.

Remark The reflections S_γ do not change the length of the vector x on which they act.

This theorem allows us to prove that the system

$$\pi_{\mathfrak{g}} \equiv \{\alpha_1, \dots, \alpha_r\} \tag{9}$$

of simple roots of $\mathfrak{g}$ defines uniquely Δ. In other words, applying to $\pi_{\mathfrak{g}}$ all possible reflections S_{α_i} and their combinations $S_{\alpha_i} S_{\alpha_j}$, $S_{\alpha_i} S_{\alpha_j} S_{\alpha_k}$, ...etc. will give us the whole root system Δ.

Actually it is the geometrical and symmetry properties of Δ that are important, not the absolute coordinates of α_i in $\mathbb{E}^r$. So it is enough to know the relative lengths and the angles between them and the latter are fixed by the *Cartan matrix*. This is an $r \times r$ *matrix* (r is the rank of $\mathfrak{g}$) given by:

$$C_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}. \tag{10}$$

Obviously, $C_{jj} = 2$.

The simple roots of $\mathfrak{g}$ form a basis in the root space $\mathbb{E}^r$ which is not orthonormal. Alonh with them we can introduce anoter basis $\omega_1, \dots, \omega_r$, which is bi-orthonormal to the simple roots, namely ω_j are defined by:

$$\frac{2(\alpha_i, \omega_j)}{(\alpha_i, \alpha_i)} = \delta_{ij}.$$

ω_j are known as the **fundamental weights** of $\mathfrak{g}$.

Finite-dimensional representations of $\mathfrak{g}$ and weight systems.

We will construct all finite dimensional representations of $\mathfrak{g}$ in such a way that the Cartan generators will always be represented as diagonal matrices.

Let V_ω be a finite-dimensional space in which such representation is determined. We start by introducing a basis in V_ω such that the Cartan generators are diagonal. To this end we split V_ω into direct sum of subspaces, which are labelled by weights of this representation:

$$V_\omega = \bigoplus_{\gamma \in \Gamma_\omega} V_\omega^{(\gamma)}, \qquad \gamma \in \Gamma_\omega.$$

The weights and the weight system $\gamma \in \Gamma$ share some (but not all) of the properties of the root systems. The weights γ label the subspaces $V_\omega^{(\gamma)}$ which are defined as eigen-subspaces of all Cartan generators:

$$H|\gamma\rangle = \gamma(H)|\gamma\rangle \equiv (\vec{h}, \gamma)|\gamma\rangle, \qquad \text{for all} \qquad |\gamma\rangle \in V_\omega^{(\gamma)}.$$

This definition ensures that all Cartan generators are diagonal with this choice of the basis.

Of course the Weyl generators are not diagonal. We shall prove that

$$E_\alpha V_\omega^{(\gamma)} \in V_\omega^{(\gamma+\alpha)}.$$

Indeed,

$$\begin{aligned} HE_\alpha|\gamma\rangle &= ([H, E_\alpha] + E_\alpha H)\,|\gamma\rangle \\ &= (\alpha(H) + \gamma(H))\,E_\alpha|\gamma\rangle \in V_\omega^{(\gamma+\alpha)}. \end{aligned}$$

Definition $|\omega\rangle$ is the highest weight vector in V_ω if:

$$E_\alpha|\omega\rangle = 0 \qquad \text{for all} \qquad \alpha > 0.$$

$|\omega\rangle$ is the analog of $\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$. All upper-triangular matrices (like E_α for positive α) acting on it give 0.

Theorem. If the representation is irreducible then $\dim V_\omega^{(\omega)} = 1$ and vice versa.